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A NEW APPROACH TO POTENTIAL FLOW AROUND AXISYMMETRIC BODIES

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SUMMARY

A new method is introduced to solve potential flow problems around axisymmetric bodies. The approach relies on expressing the infinite series expansion of the Laplace equation solution in terms of a finite sum which preserves the Laplace solution for the potential function under a Neumann-type boundary condition. Then the coefficients of the finite sum are calculated in a least squares approximation sense using the Gram–Schmidt orthonormalization method. Sample benchmark problems are presented and discussed in some detail. The solutions are accurate and converged faster when a rather small number of terms were used. The method is simple and can be easily programmed.

KEY WORDS Potential flow Axisymmetric bodies Least squares

1. INTRODUCTION

Calculation of the potential flow around axisymmetric bodies is very important since most of practical shapes are axisymmetric or nearly so, e.g. airships, submarines, torpedos, etc. Also, in viscous flow situations the solution of the boundary layer equations requires the pressure distribution outside the boundary layer, which can be precisely calculated from the potential flow solution. In addition, the actual lift can be calculated with reasonable accuracy by solving the potential flow, unless the flow is separated.

In this study a new method has been introduced for the solution of potential problems around axisymmetric bodies. In two-dimensional potential flow situations both the streamfunction and potential function satisfy the Laplace equation and can therefore be treated entirely by complex variable methods. Although axisymmetric bodies involve only two variables, only the potential function satisfies the Laplace equation. Therefore the flow field is not Laplacian and the method of employing analytic functions of complex variables cannot be used.

One of the best known methods for potential flows around axisymmetric bodies was developed by von Karman[1] in which sources/sinks with constant strength for each element are distributed along the axis of the body. The strengths of the elements are determined from the condition of zero streamfunction over the body. However, in this method the resulting system of linear equations is generally ill-conditioned. Zedan and Dalton[2,3] developed the von Karman method by varying the strength of the sources linearly for each element. Their method converged faster and produced more stable and accurate results than von Karman's.

Another approach is the surface singularity distribution. The use of sources distributed over the surface of the body leads to an integral equation of Fredholm's second kind for which a solution

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always exists. Comprehensive reviews of the solution of potential flow problems using the surface singularity method are given by Hess.[4,5]

James[6] developed a general analytical method for axisymmetric potential flows. In his method the flow field is represented as a sequence of analytical functions (Fourier, Chebyshev, Legendre, etc.).

Campbell[7] combined axial source and least squares methods for calculating potential flows over simple surfaces. Dasgupta[8] used a finite element method in the solution of potential flows. The exterior region was confined to an infinite assembly of geometrically similar finite element cells. A technique which uses a substructuring scheme over an infinite collection of finite elements was described.

2. PROBLEM STATEMENT AND THE METHOD

It is known that the potential function satisfies the Laplace equation for potential flow over bodies:

$$\nabla^2\phi=0 \quad \text{in } D. \tag{1}$$

The normal component of the fluid velocity must vanish on the impervious surface of the boundary. Thus

$$\frac{\partial\phi}{\partial n}=\text{grad } \phi\cdot\mathbf{n}=0, \tag{2}$$

where $\mathbf{n}$ is the unit outward normal vector of the surface. A regularity condition at infinity is

$$|\text{grad } \phi|\rightarrow 0. \tag{3}$$

Since the bodies taken into consideration are axisymmetric, the potential function has two variables (r, θ). The potential function can be represented by a combination of those of uniform flow and a disturbance function $T(r, \theta)$ such that

$$\phi(r, \theta)=Ur\cos\theta+T(r, \theta), \tag{4}$$

where U is the upstream fluid velocity. Therefore equations (1) and (2) take the form

$$\nabla^2 T(r, \theta)=0 \quad \text{in } D, \tag{5}$$

$$\frac{\partial T(r, \theta)}{\partial n}=-\frac{\partial(Ur\cos\theta)}{\partial n} \quad \text{on } S. \tag{6}$$

The general solution of the Laplace equation for the exterior region of an axisymmetric body is given as[9]

$$T(r, \theta)=\sum_{n=0}^{\infty}\frac{A_n}{r^{n+1}}P_n(\cos\theta) \quad (0<\theta<\pi), \tag{7}$$

where P_n are Legendre polynomials of order n and A_n are coefficients to be determined.

Here we introduce our method in which we propose the solution to be of the form

$$T(r, \theta)=\sum_{n=0}^{M}D_nF_n(r, \theta), \tag{8}$$

where D_n are coefficients to be determined and $F_n(r, \theta)$ is taken as

$$F_n(r, \theta)=\int_{-L}^{L}\frac{f^n\,\mathrm{d}f}{\sqrt{(f^2-2rf\cos\theta+r^2)}}. \tag{9}$$

It should be noted that with the help of

$$\frac{1}{\sqrt{(1-2R\cos\theta+R^2)}}=\sum_{k=0}^{\infty}\frac{P_k(\cos\theta)}{R^{k+1}}, \tag{10}$$

$F_n(r, \theta)$ has the same form as equation (8). Equation (9) can be analytically integrated easily. If this is done, a recursive relation can also be obtained. Therefore, if we let

$$R(f)=\sqrt{(f^2-2rf\cos\theta+r^2)},$$

$$F_0(r,\theta)=\ln\left(\frac{R(L)+L+r\cos\theta}{R(-L)-L+r\cos\theta}\right), \tag{11}$$

$$F_1(r,\theta)=R(L)-R(-L)-r\cos\theta\, F_0(r,\theta), \tag{12}$$

$$F_m(r,\theta)=\frac{1}{m}[L^{m-1}R(L)-(-L)^{m-1}R(-L)-(2m-1)r\cos\theta\, F_{m-1}(r,\theta)$$
$$-(m-1)r^2F_{m-2}(r,\theta)],\quad m\geqslant 2. \tag{13}$$

If we let

$$\psi_n(r_{\rm B},\theta)=\left(\frac{\partial F_n}{\partial r}\frac{\partial\Omega}{\partial r}+\frac{1}{r^2}\frac{\partial F_n}{\partial\theta}\frac{\partial\Omega}{\partial\theta}\right)_{r=r_{\rm B}}, \tag{14}$$

where $\Omega=\Omega(r,\theta)$ is the surface equation of the body and $r_{\rm B}$ is the radial vector of the surface of the body, then the boundary condition can be expressed as

$$\Pi(\theta)=\sum_{n=0}^{M}D_n\psi_n(r_{\rm B},\theta)=\left(-U\cos\theta\frac{\partial\Omega}{\partial r}+\frac{U\sin\theta}{r}\frac{\partial\Omega}{\partial\theta}\right)_{r=r_{\rm B}}. \tag{15}$$

In order to find the coefficients D_n, an orthonormal set of linear combinations of functions $\psi_0, \psi_1, \ldots, \psi_n, \ldots$ is to be produced. The Gram–Schmidt orthonormalization process is employed:[10]

$$\gamma_0=k_0\psi_0,$$
$$\gamma_1=k_1(\psi_1-\alpha_{1,0}\psi_0),$$
$$\gamma_2=k_2(\psi_2-\alpha_{2,0}\psi_0-\alpha_{2,1}\psi_1),$$
$$\gamma_n=k_n\left(\psi_n-\sum_{i=0}^{n-1}\alpha_{n,i}\psi_i\right), \tag{16}$$

such that the set of $\gamma_0, \gamma_1, \gamma_2, \ldots, \gamma_n$ is orthonormal, i.e.

$$\int_0^{\pi}\gamma_n(\theta)\gamma_m(\theta)\,{\rm d}\theta=\delta_{nm}, \tag{17}$$

where δ_{nm} is the Kronecker delta function.

The constants $k_0, k_1, \ldots$ and $\alpha_{1,0}, \alpha_{2,0}, \alpha_{2,1}, \ldots$ are determined by the orthonormalization process such that

$$\alpha_{n,m}=\int_0^{\pi}\psi_n(\theta)\gamma_m(\theta)\,{\rm d}\theta, \tag{18}$$

$$k_n=\left(\int_0^{\pi}\psi_n^2(\theta)\,{\rm d}\theta-\sum_{i=0}^{n-1}\alpha_{n,i}^2\right)^{-1/2}. \tag{19}$$

Since the orthonormal set of the functions $\gamma_0, \gamma_1, \gamma_2, \ldots$ is the linear combination of $\psi_0, \psi_1, \psi_2, \ldots$, equation (15) can be written

$$\Pi(\theta) = \sum_{n=0}^{M} D_n \psi_n(\theta) = \sum_{n=0}^{M} A_n \gamma_n(\theta). \tag{20}$$

Since the ψ_n are orthonormal, the coefficients A_n can be determined easily from

$$A_n = \int_0^{\pi} \Pi(\theta)\gamma_n(\theta)\,d\theta. \tag{21}$$

The coefficients D_n are found from the relation

$$D_n = \sum_{i=n}^{M} A_i a_{in}, \tag{22}$$

where the a_{in} are found from

$$a_{in} = \begin{cases} k_i, & \text{if } i = n, \\ -k_i \sum_{l=n}^{i-1} \alpha_{i,l} a_{ln}, & \text{if } i \neq n, \quad i > n. \end{cases} \tag{23}$$

3. RESULTS AND DISCUSSION

The study includes four benchmark problems. The first three are ellipsoids of revolution with slimness ratios (b/a) of 0·5, 0·25 and 0·125. These benchmark problems are recognized to be a severe test of very slender bodies; also, the exact solutions are available. The last benchmark problem is a slender body (profile geometry F-57[3]).

In this study the effect of L on the solution and the number of terms to be used were analysed. In Figures 1 and 2 the effect of L on ellipsoids of revolution with slimness ratios of 0·5 and 0·25 is depicted respectively. As can be seen, the effect of L in Figure 1 is less pronounced. One has the

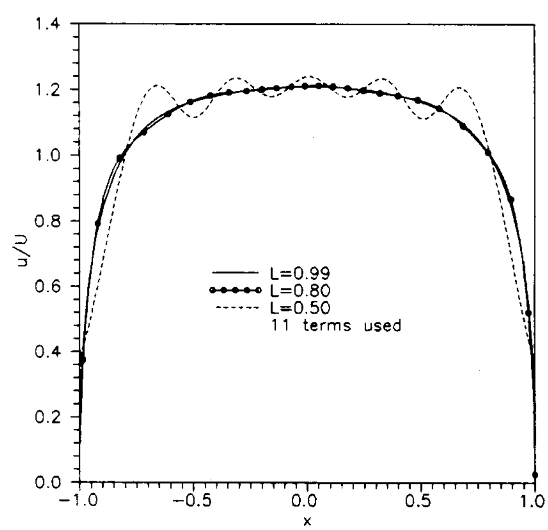

Figure 1. Effect of the parameter L on the velocity profile for a slimness ratio of 0·5

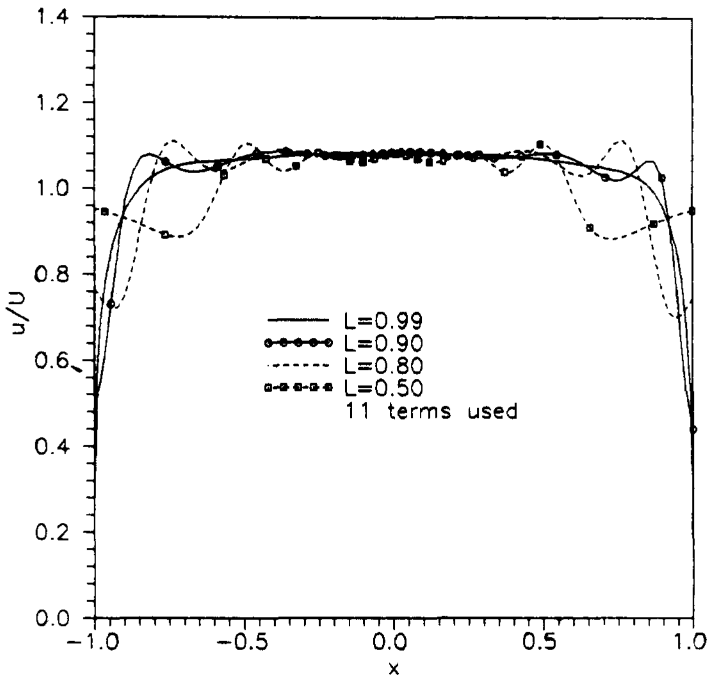

Figure 2. Effect of the parameter L on the velocity profile for a slimness ratio of 0·25

freedom of choosing a wide range of L-values without compromising the accuracy of the solution. However, for an L-value of 0·5 oscillations in the solution are observed. It should also be noted that the solution has converged for the number of terms used in both cases (11 terms). In Figure 2 the slimness ratio is smaller and poses an even more severe test on the effect of the parameter L. Here it is seen that the freedom in choosing the L-value is more restricted. When the L-value is chosen close to unity (in this case), the solution converges well to the exact solution. For example, even for an L-value of 0·9 in Figure 2 severe oscillations are observed. In our benchmarks we scaled the geometries to the range $-1 \leqslant x \leqslant 1$, but we must state that $F_n(r, \theta)$ can also be chosen as

$$F_n(r, \theta) = \int_a^b \frac{f^n \, \mathrm{d}f}{\sqrt{(f^2 - 2rf\cos\theta + r^2)}}, \tag{24}$$

where a and b are the lower and upper x-limits of the body.

In Figure 3 the solutions of the ellipsoid of revolution benchmark problems for the slimness ratios 0·5, 0·25 and 0·125 are shown. These solutions were obtained with $L = 0{\cdot}99$ and 15 terms were used. The solutions match well with the exact solutions so that the exact solutions cannot be differentiated. Thus only our solutions are displayed.

In Figure 4 the solutions for the profile geometry of body F-57 are compared. The calculated velocity distribution and the prescribed velocity distribution show very good agreement. In this problem an L-value of 0·99 and 15 terms were used. The increase in the number of terms has not improved the solution to the extent desired. The effect of the L-value is also very severe in this benchmark problem. Therefore the L-value was chosen as $L = 0{\cdot}99$. For lower values of L oscillations are observed.

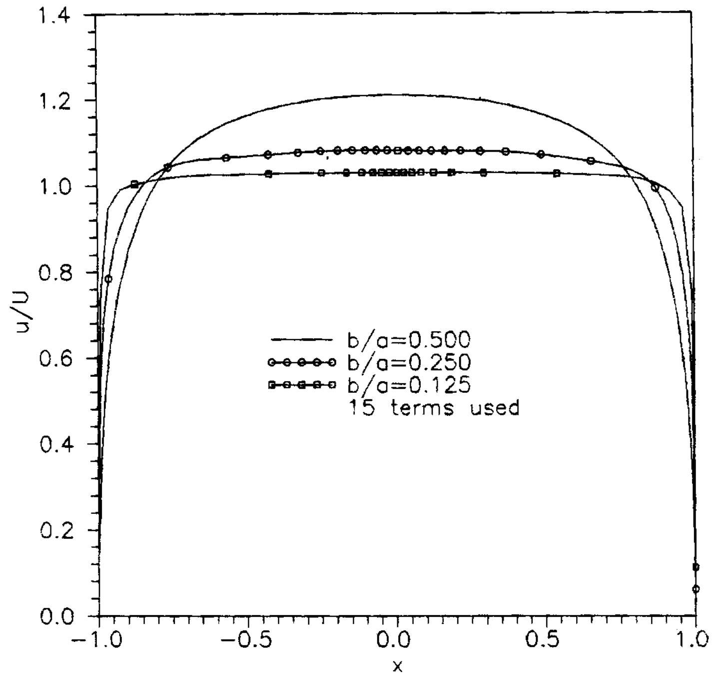

Figure 3. Velocity profiles for ellipsoids of revolution with various slimness ratios

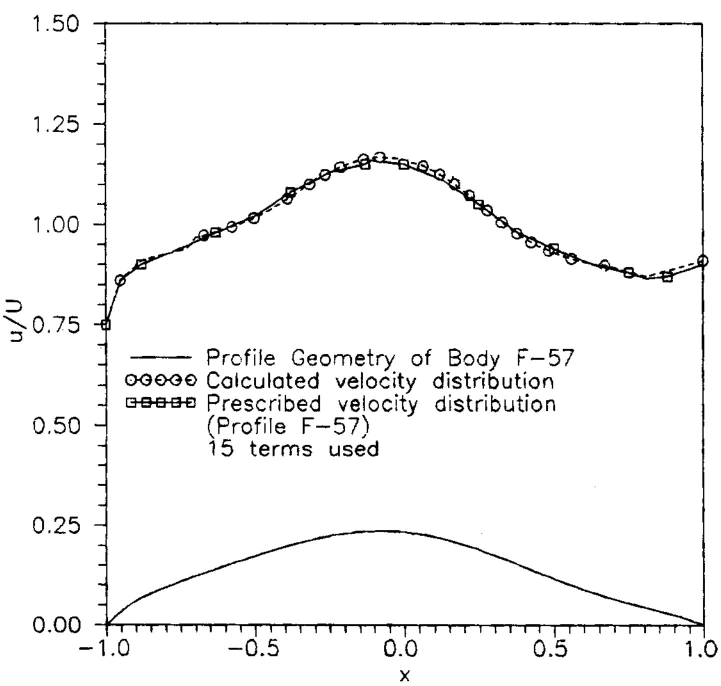

Figure 4. Velocity profile for profile geometry of body F-57

For all the problems the use of 10 or more terms gives convergence to a solution, while with a smaller number of terms the solution does not converge and displays oscillating behaviour.

4. CONCLUDING REMARKS

A new approach to the solution of potential flow around axisymmetric bodies is introduced. The method is very simple and easily programmable. The computation time with this method is much less than with other available methods. The computation time for the benchmark problems solved was of the order of a few seconds with an IBM PC-XT compatible machine (8 MHz). The L-value when chosen very close to the tails of the bodies assures a correct solution provided that a sufficient number of terms are used. The method itself should be studied in more detail for different geometries.

APPENDIX: NOMENCLATURE

ϕ	potential function
D	domain of the body
$\mathbf{n}$	normal vector of the surface
U	upstream fluid velocity
T	disturbance function
r, θ	radial co-ordinates
S	surface of the body
P_n	Legendre functions of degree n
Ω	equation of the surface
r_B	radial vector of the surface of the body
F_n	functions defined by equation (9)
R	a dummy variable used in equation (10)
f	a dummy integration variable used in equation (9)
L	bounds of the region

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